Computational Music Theory

PORTE AKADEMİK
MÜZİK VE DANS ARAŞTIRMALARI DERGİSİ
JOURNAL OF MUSIC AND DANCE STUDIES

Araştırma Makalesi / *Research Article*

Computational Music Theory

Hesaplamalı Müzik Teorisi

Ruhan ALPAYDIN[1]

(1) Sorumlu Yazar / Corresponding Author:
Ruhan Alpaydın, İTÜ MİAM Doktora Öğrencisi
E-posta: ruhan.alpaydin@gmail.com
ORCID: 0000-0002-2472-9563

Atıf / Citation: Alpaydın, R.(2022)
Computational Music Theory.
Porte Akademik Müzik ve Dans Araştırmaları Dergisi,
22,7-22.

Abstract

This article is an introduction to computational music theory. Music theory examines the structure of music at the neutral level through its dimensions: melody, harmony, rhythm and meter, timbre, texture and form. Computational music theory uses the language of mathematics, algorithms and computational power to examine music. This examination includes both static modelling of musical structures and dynamic modelling of musical processes. The use of computational power in music theory —not at the sound level, but at the symbolic level, i.e., at the level of notes and higher abstractions— is relatively new. MIDI, MusicXML and similar formats have become popular forms of musical information exchange for music at symbolic level. We define "computability" and examine music theoretical concepts relevant to analytical processing of musical information. We review recent mathematical and computational models for music. These models include the Rubato line of research, i.e., mathematical music theory which has been in continuous development since 1980's. The example we will examine in detail is both a mathematical and a computational model for analysis of harmony and meter and associated software implementations on Rubato, a Java-based music composition and analysis framework. Results show that detailed experiments on musical information is possible for testing various theses about music theory via computational modelling of musical information.

Keywords: Music Theory, Rhythm, Metric Analysis, Harmonic Analysis, Rubato.

1. INTRODUCTION

Music's ontologies are its *mental*, *physical* and *psychological* realities. Music is also *communication*, i.e., it has *poiesis*, *neutral* and *aesthesis* levels which are related to the semiotical and gestural/embodiment aspects of the communication (Mazzola, 2011). Musical creation at the *poiesis* level begins as ideas at the mental level (i.e., in our inner world) and the musical score is a representation of these ideas as a composition. The score is a symbolic representation or an abstraction of the music in performance. The actual or real music is the music in the air (or any other medium), i.e., performed music or music in performance. The *aesthetic level*, i.e., perception and cognition of music covers all *analytical*, *emotional* and *semiotic* processes that are active in the listener (Mazzola, 2011).

Musicology, music history and ethnomusicology all study music as a social science. Music theory and systematic musicology are closer to considering music as an exact science in terms of the methodologies they use (Parncutt, 2008). For both, but in particular for the latter, i.e., for music theory and systematic musicology, musical objects and musical processes should be defined precisely. The language of mathematics can be used to bring this exactness to the definition of music theory terms and concepts by considering musical objects as mathematical structures. If structures in music can be considered as structures that are already studied in mathematics, then properties of the mathematical structures are also properties of the musical objects. This is the case, for example, in mathematical physics. In physics, such a consideration results in understanding properties of physical objects in the micro and macro world. The interaction is also the other way around. For example, only after the invention of calculus by Newton and Leibniz, mechanical processes such as motion and gravity could be modelled (Edwards, 1982). Consequently, mathematical structures have been used to discover structural properties of music as a physical reality at the neutral level in symbolic form. Forte (1977) used set theory to analyze atonal music. Lewin (1982) used group theory for analysis of melodic and harmonic language. Temperley (2002) used probability theory for music analysis. Through Mazzola's (2002) formula for inner meter evaluation, Volk (2009) computed the metric language of various music genres. Symbolic representations of music at the neutral level such as MIDI or MusicXML can serve as objects of empirical experiments. Such analyses bring invaluable insights into the focus of music theory: melodies and motifs (Frieler, 2017), metric and rhythmic organization (Mazzola et al., 2020), harmonic language (Noll and Garbers, 2001; Alpaydin and Mazzola, 2015), texture, timbre and form.

Numerous scientific disciplines have deployed computational power for rapid processing of digitized information. Recent research areas, to name a few, are computational biology, computational material science, and computational epidemiology. In fact, computational power has become an indispensable tool for almost all fields of science because computers can process information with speed, precision and accuracy —i.e., free of execution error. Some example domains where processing of vast information is possible only with computational power are genomic analysis, population dynamics and computational material design. Still, a computer manipulates information according to a procedure —more precisely, an *algorithm*— which is designed by a human. Rapid processing of vast data may result in interesting outcomes that may not be possible, for example, by hand analysis within a researcher's lifetime. Moreover, very detailed and sophisticated computational analysis on data may discover patterns and bring out useful results that may not immediately be recognizable by human perception and cognition. Musical patterns which can only be discovered by an adequate modelling and a thorough computational analysis may still have an aesthetic appeal and can be felt; but they may not be described through music theoretical terminology only. Consequently, music theory and musicology research has been using computational power at an increasing rate, and research fields like Music Information Retrieval (Schedl et al., 2014), Computational Ethnomusicology (Tzanetakis et al., 2007), and Mathematical Music Theory (Mazzola, 2002) are a few of the relatively new research directions with methodologies that use mathematics, statistics, and computation.

In the next section, we will explain computation including its power and limitations. Then, we will review history of mathematical and computational modelling in music theory. Our domain of operation is Western music of the common practice period (Baroque, Classical and Romantic periods of European music) because historically, the greatest scientific endeavour has been used to study these musical genres. However, our ideas can be extended to other genres or music of other cultures. Afterwards, we will give one research example: mathematical and computational analyses of harmony and meter. We will overview the computational models for perception/cognition of harmony and meter. Then, we will review how a metric hierarchy of musical events can be integrated into the perception of harmony. All computational models are implemented as plug-in components on the Java-based Rubato music composition and analysis framework (Milmeister, 2007). These components can also be considered as reincarnations of HarmoRUBETTE and MetroRUBETTE, which were previously designed and programmed on Rubato (Mazzola, 2002).

2. HOW CAN COMPUTATION BE USEFUL FOR MUSIC THEORY?

Computability or computable functions refer to the same phenomenon which was conceived mathematically in the 1930s. Computation refers to a sequence of logical operations on symbols which are stored in computer memory (Boolos et. al, 1974). Early predecessors of computation devices include computers that were used in symbolic operations required to break cryptosystems that were used for secret communication during World War II. Coding theory, information theory and computational theory were developed together out of these cryptoanalytic research efforts (Hilton, 1984). However, computer science as a scientific field also arose from developments in mathematical logic in the 20th century: recursive functions proved to be equivalent to computable functions (Boolos et al., 1974). More practically, a computable function is a mathematical function which can be "computed'' on the Turing-Post Machine[1] (TPM), which is an abstract machine[2]. A computation takes place on the TPM which has a tape of cells corresponding to an infinite linear memory organization and a program/instructions of finite length. The instructions of the program can write symbols (from a finite alphabet) on these cells and read symbols from the cells, going back and forth over the tape. The TPM is equivalent in computational power to the universal digital computers of today. What a TPM can compute is the true meaning of computation. For Turing (1939), "A function is effectively calculable if its values can be found by some purely mechanical process". If we can compute a function with a TPM, then we can write an equivalent program on a computer.

Computation and music theory have met in two ways. Since the 1950s, with the appearance of digital computers, sound and musical information have become increasingly digital. Secondly, as is the case for other digitized/quantized information, computational power has been used to process musical information with precision, accuracy and speed that had not existed before. This is true both for music at symbolic level and music at the audio (signal) level.

Music theory deals with local and global melodic, harmonic, and metric/rhythmic structures as well as timbre and textural organization (Milmeister, 2007, pp. 3-6). It also deals with more ontological dimensions like time, sound properties, and pitch systems, including tuning and temperament. Moreover, embodiment, gesture theory, and musical semiotics have been active research areas that are connected to music theory and that extend it. Traditional score-based musical notation as well as representation of music at symbolic or audio levels in digital format is also part of music theory (Milmeister, 2007, pp. 7-18). In modern music of the 20th century, a special emphasis is given on timbre and textural structures by composers and therefore, theory extends to these domains as well. Finally, in music theory, *form* traditionally refers to the local and global organizations of music that include all dimensions, i.e., melody, harmony, meter/rhythm and timbre/texture (Caplin, 1998). With its concepts, tools and methodologies and theories, such as the Schenkerian Analysis or Set Theory, music theory helps analyze music and create music. In our geography, such theoretical inquiries began with the Sumerians and Old Babylonians who wrote instructions for tuning stringed instruments in the first music theory books, i.e., clay tablets (Burkholder, 2014, pp. 6-8).

Music, when studied as science, has the musical material in front of the scientist as music at the 'neutral level'. Musical information at the note level and above in almost all dimensions is discrete[3]. For example, a *pitch system* with non-zero period *oct* is a finite set S of pitch-classes that is oct-periodic. Keys on a piano comprise a pitch system and the period is the octave. All of the white keys on the piano amount to another pitch system with the same period, the octave. *Scale, mode, chord, harmonic function, and tonality* also have precise mathematical definitions (Alpaydin and Mazzola, 2015, pp. 8-9). On the time domain, *meter* and *rhythm* turn a

1 Turing Machine (by Alan Turing) and a Post Machine (by Emil Post) are equivalent in computational power and both were conceived in the 1930s.
2 Turing Machine does not and need not physically exist. Simulations as software (Turing Machine Visualisation, 2022) and hardware exist (Turing Machine Physical Realization, 2022).
3 We consider music at a symbolic level, i.e., note level and above in this article.

continuous[4] time domain into a discrete domain via periodic *beats*. Discreteness of musical information also results in effortless and lossless encoding, modelling and processing. An example is the Musical Instrument Digital Interface (MIDI), which is an encoding of musical information for exchange between digital instruments. Thus, we see that, as early as 1980s, quantization of musical information appeared as a practical necessity. Today, in the internet age, MIDI is a widely used popular format for music information exchange at symbolic level (Milmeister, 2007, pp. 13).

3. MATHEMATICAL AND COMPUTATIONAL MODELLING OF MUSIC

Computational models of music are built on mathematical models of musical objects and processes. Music theory begins with the construction of the first musical scales, modes, and instruments, from which the first melodies were created (Burkholder, 2014). The first rhythms arising from the first percussive instruments (which may have percussive function as a side effect) may have appeared earlier than these scale systems because percussive movements on objects are more ubiquitous compared to pitched music instruments.

Advances in science and technology inevitably resulted in new instruments as well as evolvement of the existing musical instruments in different directions. Examples are fortepiano, saxophone, theremin and electric guitars. Music theory and musical instrument design have mutually influenced each other and have co-developed. For example, history of tuning and temperament in European music from Pythagorian tuning to 12-tone equal temperament is a result of such a co-development (Bibby, 2006). Simple mathematics for calculation of intervals in scale systems as well as construction of instruments have existed since antiquity (Burkholder, 2014). However, using advanced mathematical concepts for music theory is relatively new. Modern well-known contributions from science to music theory came from linguistics (Lerdahl and Jackendoff, 1983) and mathematics (Lewin, 1982). Yet, in 1928, a young mathematician, Wolfgang Graeser, had already analyzed Bach's Art of Fugue with mathematical group theory. The European School of Mathematical Music Theory resulted in fruitful new directions in the last 40 years in terms of theoretical research as well as development of practical tools for experimentation on music at the symbolic level (Mazzola, 2002).

It should be noted that, in the USA, the music and computer technology relationship started as early as the development of the computer technology itself, i.e., in the 1950s and 1960s, and formal languages were designed for music symbolism in research institutions. MUSIC-N family of music languages were initiated by Max Mathews, a researcher in Bell Labs in 1950s (Roads and Mathews, 1980). In the same decade, ILLIAC I suite by Hiller and Isaacson on the ILLIAC I computer were used to compose a four part suite, selecting musical material based on constraints such as counterpoint rules (Hiller, 1957). In fact, this was an example of what was understood as “Computer Music” in those times, music which is composed by a computer at a symbolic level.

Another milestone in music modeling is Lerdahl and Jackendoff's (1983) Generative Theory of Tonal Music (GTTM). GTTM is based on work of a linguist and a composer who describe a grammar to “parse'' tonal music. A grammar is a finite set of rules that can generate an infinite number of sentences as in the case of a natural language grammar or other formal grammars such as programming languages. GTTM's grammar is made up of a looser set of rules compared to the formal definition of a grammar (Sipser, 2006, pp. 100-104). GTTM's rules are preference rules and well-formedness rules that help understand the hierarchical structure of tonal music. Rules are based on Gestalt Theory and music psychology. Similar to our grouping a sequence of words into a noun phrase or a sentence, group of pitches or onsets are grouped into higher abstractions

4 'Continuous' is the antonym of 'discrete'. Time is a continuous dimension but its digitization/quantization into seconds, miliseconds, etc. turns it into a discrete dimension. Similarly, frets on the guitar turn a continuous frequency domain into a discrete domain but on a fretless guitar or a cello, the pitch dimension is continuous.

when we hear tonal music. As the authors Lerdahl and Jackendoff (1983) also underline, GTTM's purpose is not generation of new musical sentences, perception and cognition of tonal music, i.e., how we group and create a hierarchy of musical events when we hear music.

Lewin's (1982) work begins from Riemann's ideas. He explains the 12-tone equal tempered scale as a cyclic group structure with pitch transformations. Lewin grounds his transformational theory on *group theory* observing that both m (from 'mediant') and d (from 'dominant') are transformations on pitch-class and are the generators of the 12-tone equal tempered scale. This scale is the Z_{12} (also shown as Z/12) cyclic group which is the set {0,1,2,…11} corresponding to 12 pitch-classes such that 0=C, 1=C#/Bb, etc. Construction of the diatonic scale using dominant transformation beginning from F is FCGDAEB and the continuation would give the whole 12-tone scale FCGDAEBf#c#g#d#a#. Based on Riemann's Tonnetz —a geometric representation of pitch and triad relationships—, Riemann names these structures as Riemann Systems. Using the mediant transformation, the diatonic scale would be formed as FACEGBD. Now, Lewin makes a further abstraction to m and d, stating that m is not necessarily a transformation of a third interval which leads to other Riemann Systems. For example, taking m as the fourth interval would again generate the diatonic scale CFBEADG. A Riemann System = (T, m, d) is an ordered triple such that T is a pitch-class and m, d are intervals d≠0, m≠0 and d≠m. Then, he defines transformations on Riemann Systems and proves that four different transformations on Riemann Systems also form a group (Lewin, 1982, pp. 38). He then uses this systemic view to explain how music works through musical passages of Bach, Beethoven, Brahms, Stravinsky, Wagner, and Rachmaninoff. Through this effort, he is able to explain musical organization as transformation of pitch material using a comprehensive model: The Riemann System. He had used transformational theory to analyze atonal works as well (Lewin, 1993). More generally, wherever there is a symmetry in form (or structure), there is a mathematical group; i.e., a set, a binary operation, and four properties or restrictions about how the binary operator and the set work together: closure and associativity of the operation; a single identity element in the group and each element having an inverse element. The relation between symmetry in art forms and group theory has fascinated scientists. Using group theory, Weyl (1983) explains various types of symmetries in visual arts and craftwork since the Sumerians.

Continental Europe has been the cradle of not only the evolution of Western art music but also scientific thinking and enlightenment: Zarlino, Rameau, Riemann, and Helmholtz, who shaped music theory along centuries, are all from this tradition. IRCAM, the Institute for Research and Coordination in Acoustics/Music in Paris, has hosted important scientific research in music since the 1970s. Similar to David Lewin and Milton Babbitt, Pierre Boulez, founder of IRCAM, is a figure who had a mathematical training first and then moved into music research, using his analytical background for music composition and theory/analysis. IRCAM has hosted research in acoustics, audio/signal processing, music informatics and music composition, including electroacoustic music. IRCAM's primary importance has been uniting artistic and scientific research since its inception and this approach has led to tremendous creativity in music research and contemporary music (Di Guigno, 1986; Hirschberg, 1996; Bresson et al., 2010; Andreatta et al., 2013). OpenMusic is a Lisp-based visual composition software and has been in continuous development at IRCAM since the late 1990's (Agon et al., 1999).

The Zurich School of Mathematical Music Theory considered musical structures using advanced mathematics that went beyond group theory such as category theory and topos theory (Mazzola, 1985). Music theory, performance, composition and analysis are described in depth using a mathematical language in Topos of Music. The book's accompanying software, Rubato Composer, is a demonstration or realization of this mathematical approach (Mazzola, 2002). The Rubato Composer Software is based on a category of modules which enables musical representation on mathematical structures instead of data structures provided by the underlying programming language environment. The four important analytical components developed on Rubato are metric analysis, melodic/motivic analysis, harmonic analysis, and performance analysis tools. Anja Volk (2009) worked extensively on the metric analysis software, MetroRUBETTE. Noll (1997) worked on HarmoRUBETTE, i.e., harmonic analysis tool of RUBATO. Mathematical Music Theory research spread to the American continent as well (Montiel, 2011; Fiore and Noll, 2018; Octavia and Mazzola, 2019). A recent work on the Rubato environment by Thallman (2014) is both a theoretical framework and its realization as software for gestural composition: musical creation using gestures.

4. A COMPUTATIONAL MUSIC THEORY RESEARCH EXAMPLE: ANALYSES OF HARMONY AND METER

When a musical process can be modelled as an algorithm, the process can be understood more objectively and concretely as opposed to a vague and subjective consideration. The process can be the perception and cognition of music or its synthesis, i.e., creation of music. Such an algorithm, i.e., well-defined, finite series of steps of execution on musical data (i.e., a TPM program), is what we mean by a computational model.

We will explain how mathematics and computation can be useful to analyze musical structure on two different dimensions of music: harmony and meter. Our mathematical model of harmony is based on the ideas of Rameau for considering the chords as stack of thirds and Riemann's concept of functional harmony. Our harmonic analysis algorithm completes Riemann's program, assigning a function to *any chord*, not just triads or sevenths. Keeping the Riemann Matrix abstract allows the user to define its dimensions, allowing not only tonic, subdominant and dominant functions, and not only major and minor modes. This is one of the improvements over the previous versions of HarmoRUBETTE (Mazzola, 2002; Noll, 2001).

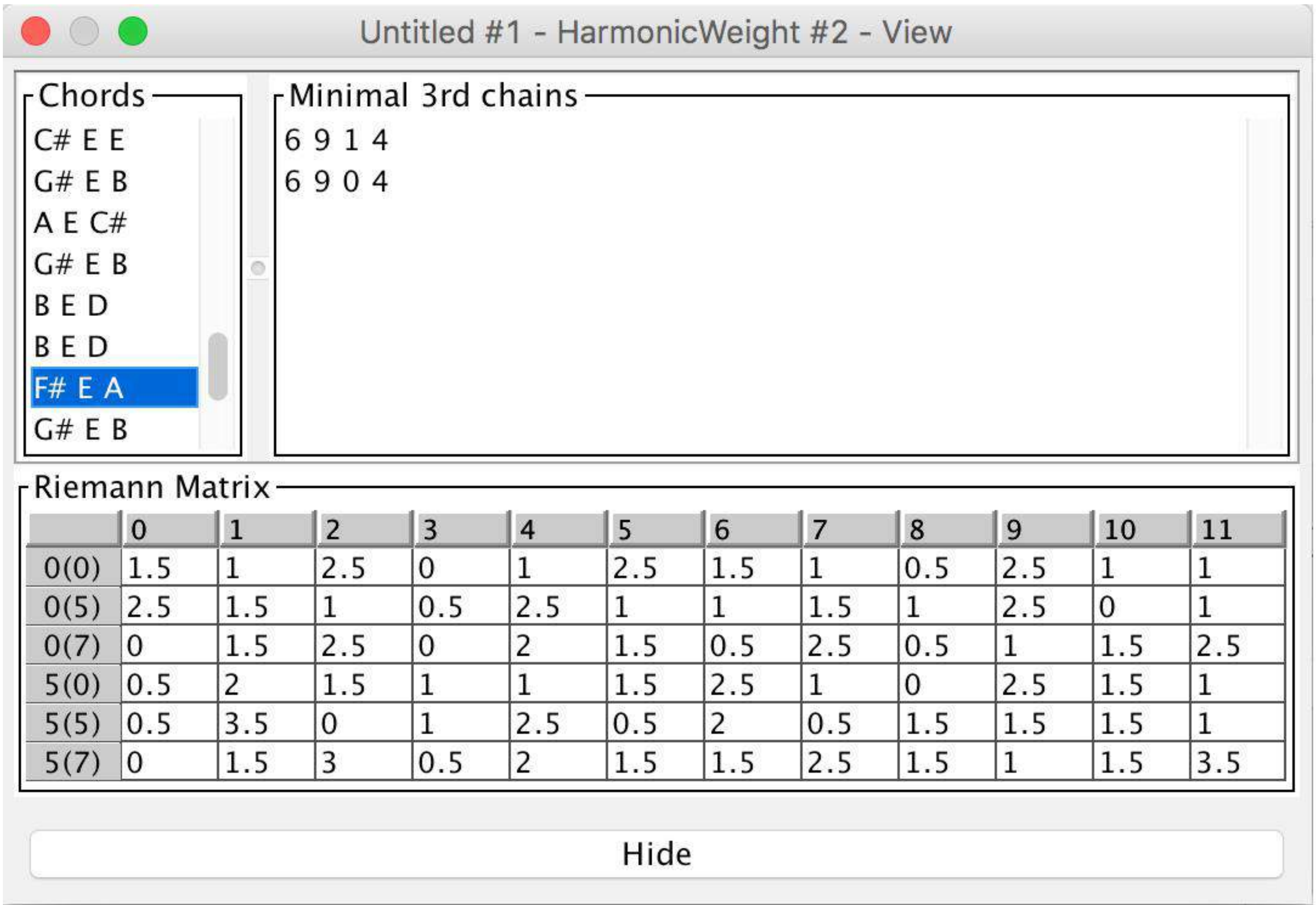
4.1. Analysis of Harmony

Harmony-based polyphony is one important feature of European music. Tonality is described by a set of pitch-classes from the 12-tone equal tempered scale where the tonic has a central importance and is the most stable, and the other pitches having particular attraction or repulsion, asking for a resolution; and giving a sense of suspension or tension of varying degrees that is always relative to this central stable pitch which is often described with the metaphor 'coming back home.' These individual pitch dynamics also hold for chords in that, with respect to an underlying tonality, the same chord might have different functions. For any tonality, the three most important chords are the tonic (I), dominant (V) and subdominant triads (IV), having the tonic, dominant and subdominant functions. Tonality is part of the content: it is a high level feature or an emerging property we hear because it is formed by musical events; but it is also a context because the underlying tonality gives meaning to the musical events.

Harmonic structure of tonal music has not only a statistical aspect, i.e. one can guess a key/tonality looking at the distribution of pitch-classes , but also a linear aspect: a *predominant* *dominant* *tonal* harmonic motion is prevalent in tonal music and this is similar to grammatical units such as a phrase or a sentence in natural languages. Statistical approaches to tonality/key recognition uses key profiles which are distributions of probabilities for each pitch-class vs. key (Krumhansl, 1990; Temperley, 1997). Key finding and music segmentation based on key regions use key profiles and bayesian methods (Temperley, 2002). Perception of harmony involves both statistical and grammatical aspects because out-of-context pitches are recognized instantly by non-musicians regardless of the ordering. An expectancy (of pitch-classes, chords, chordal progression and key change) is created by listeners based on the internalized grammar of the genre that the listener has been exposed to. Tonal harmony is such a structural determinant of the common practice period music that forms such as sonata are structured primarily based on tonality/key changes and can be segmented accordingly.

Analysis of harmony in tonal music by a human involves both mechanical processes at a local level such as labelling vertical chords and a holistic approach such as finding periods, sentences, and cadences. From these moving to individual chords, finding tonal regions, passing and neighbouring tones, and embellishing and structural chords follow (Alpaydin and Mazzola, 2015). In other words, both top-down and bottom-up examination processes exist during the analysis. There are also a number of general observations about the common practice period music:

(1) Chords are usually a stack of thirds or its inversion.
(2) Tonal tension increases or decreases between harmonic functions and also between tonalities in proportion to the distance in cycle of fifths.
(3) A metric hierarchy is imposed in musical events.
It is usually the case that overall (global) tension is low and a piece is unlikely to end in a high tension (Alpaydin, 2015).



	0	1	2	3	4	5	6	7	8	9	10	11
0(0)	1.5	1	2.5	0	1	2.5	1.5	1	0.5	2.5	1	1
0(5)	2.5	1.5	1	0.5	2.5	1	1	1.5	1	2.5	0	1
0(7)	0	1.5	2.5	0	2	1.5	0.5	2.5	0.5	1	1.5	2.5
5(0)	0.5	2	1.5	1	1	1.5	2.5	1	0	2.5	1.5	1
5(5)	0.5	3.5	0	1	2.5	0.5	2	0.5	1.5	1.5	1.5	1
5(7)	0	1.5	3	0.5	2	1.5	1.5	2.5	1.5	1	1.5	3.5

Figure 1: The chords F#-E-A and two minimal chain of thirds that the chord can be embedded into F#-A-C#-E and F#-A-C#-E are represented as pitch-class numbers. The three dimensional Riemann Matrix corresponding to this chord, constructed out of these two chains, is at the bottom (Alpaydin and Mazzola, 2015).

In this research, analysis of tonal harmony is considered as an optimization problem: a harmonic analysis that results in minimum global tension. A *chord* is a set *ch*ÌPC having the same perceived onset where PC = Z_{12}= {0,1,… 11}, i.e., 12 pitch-classes of the 12-tone equal tempered scale. Note that Z_{12} also represents 12 different tonalities.

A Riemann Matrix is assigned for every chord *ch* in the tonal piece. As we have defined above, notes in the chord may have the same onset, or may be simultaneous without having the same onset, i.e., having the same perceived onset. In Figure 1, the F#-E-A chord and its associated Riemann Matrix are depicted. The computation of the Riemann Matrix for the chord *ch* depends on a number of parameters provided by the user. The chord *ch* is embedded into at least one minimal chain of thirds. Let C = {Z_1,…, Z_n} be the set of all minimal third chains Z_i for *ch.* An arithmetical average of all such minimal chains Z_i is evaluated for every tonality, mode, and function in the Riemann Matrix:

$$weight(Ch)_{t,m,f} = \frac{1}{n}\sum_{i=1}^{n} weight(Z_i)_{t,m,f}$$

As such, a chord's weight is a function:

$$T \times M \times F \rightarrow \mathbb{R}_{\geq 0}$$

The dimensions of the Riemann Matrix are *T*, set of tonalities, *M*, set of modes and *F*, set of functions. These dimensions are determined by the user. In Figure 1, $|T| = |Z_{12}| = 12$ where 0 stands for C tonality, 1 for C#/Db tonality, 2 for D tonality, etc. *M* = {0, 5} is the set of modes where 0 is for major and 5 is for minor mode. (0 is for major mode and 5 is for the minor mode due to their order in the list of modes (Ionian, Dorian, Phyrgian, Lydian, Mixolydian, Aeolian, Locrian). *F* = {0, 5, 7} is the set of functions where 0 is for tonic function, 5 is for subdominant and 7 is for dominant function. Thus, the Riemann Matrix for chord *ch* is a function that assigns a weight larger than or equal to 0 for every possible tonality, mode, and function for *ch* in the analysis space.

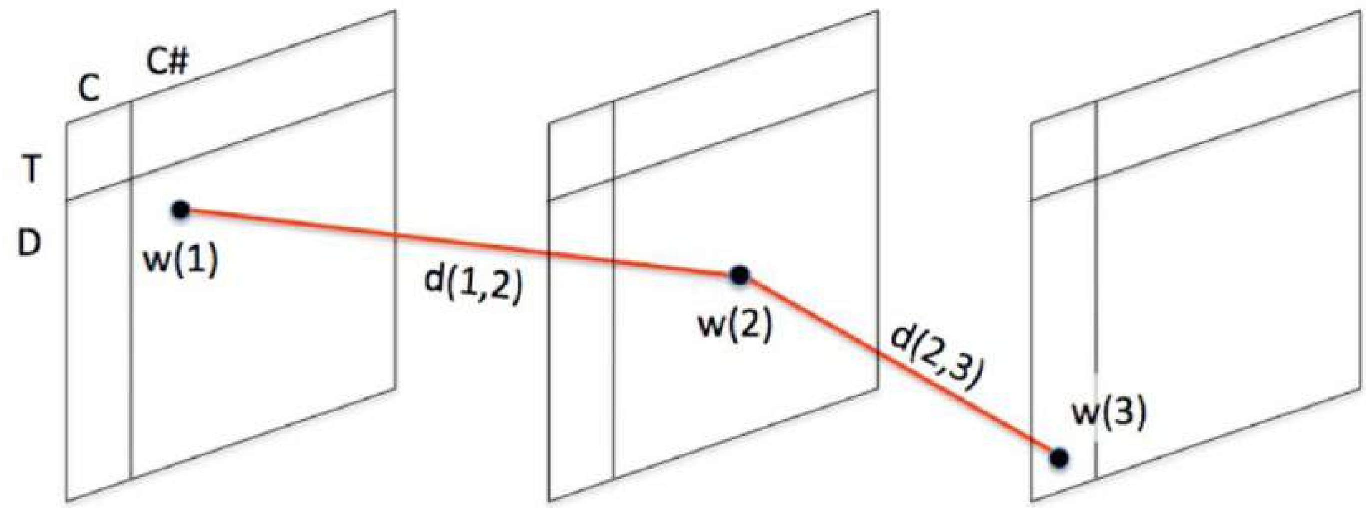


Figure 2. Three Riemann matrices in a window of size 3. A possible path is shown as a red line (Alpaydin and Mazzola, 2015).

The whole computational model for harmonic analysis is implemented as plug-in rubettes. Each rubette performs a well-defined function with an input and output interface that communicates with other rubettes as seen in Figure 3. Rubettes exchange information with one another via denotators, an internal data type of Rubato based on mathematical modules.

In Figure 3, through the Harmonic Analysis Model rubette, dimensions of the Riemann Matrix are given. These dimensions also form the analysis space. Harmonic Analysis Model rubette is also used to specify other parameters such as tension values and weight-of-thirds table (Alpaydin and Mazzola, 2015). Chord Sequencer rubette builds chords by vertically scanning the musical score. Harmonic Weight rubette builds a Riemann Matrix for each chord after finding the set of minimal chain of thirds for the chord. Harmonic Path rubette finds the optimum path among the sequence of Riemann Matrices and this optimum path is the harmonic analysis result that assigns a harmonic function to each chord. The following equation

$$weight_{(p_1,\ldots,p_n)} = \sum_{i=1}^{n} weight(p_i) + \sum_{i=1}^{n-1} distance(p_i, p_{i+1})$$

is the weight of the path $p_1, p_2, \ldots p_n$ where p_is are points in adjacent Riemann Matrices, where

$$distance(p_i, p_j) = e^{-[TonalityChange(ton_i, ton_j) + ModeChange(mod_i, mod_j) + FunctionChange(fun_i, fun_j)]}$$

There are two contributions to the weight: the weights in the Riemann Matrix and the distance between adjacent Riemann Matrices which is evaluated with respect to the change in mode, function, and tonality. These tension values are also provided by the user as table values in the Harmonic Analysis Model rubette (Alpaydin and Mazzola, 2015). The Lilypond File Out rubette converts the harmonic analysis table into score notation, a more visually pleasing form for musicians. An output of the Harmonic Path rubette can be seen in Figure 4.

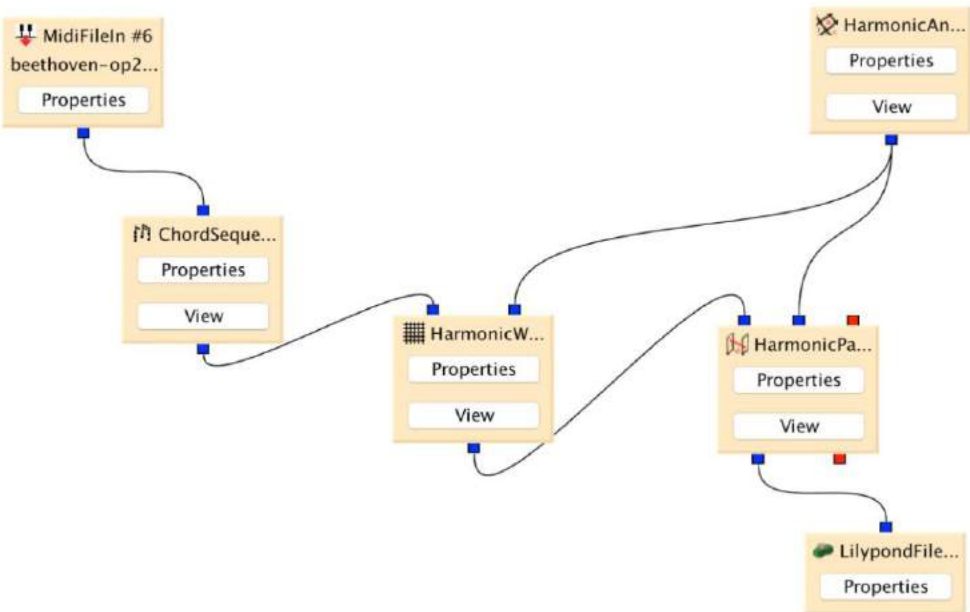


Figure 3: The network of rubettes to analyze harmony (Alpaydin and Mazzola, 2015).

Untitled #1 - HarmonicPath #3 - View

Harmonic Weights | Harmonic Path

Harmonic Analysis Result

Ch...	Onset	F	C	G	D	A	E	B	F#	C#	G#	D#	A#
1	0.00		0(0)										
2	2.00		0(0)										
3	2.25		0(0)										
4	2.50		0(0)										
5	2.75		0(0)										
6	3.00		0(7)										
7	3.50		0(0)										
8	4.00		0(7)										
9	5.00		0(0)										
10	8.00		0(7)										
11	10.00		0(0)										
12	10.25		0(7)										
13	10.50		0(0)										
14	10.75		0(7)										
15	11.00		0(0)										
16	11.50		0(7)										
17	12.00		0(0)										
18	13.00		0(0)										
19	16.00		0(0)										
20	17.00		0(0)										
21	18.00		0(7)										
22	18.25		0(0)										
23	18.50		0(7)										
24	18.75		0(0)										
25	19.00		0(7)										
26	19.50		0(7)										
27	20.00		0(0)										
28	21.00		0(0)										
29	22.00		0(7)										
30	22.25		0(0)										
31	22.50		0(7)										
32	22.75		0(0)										
33	23.00		0(7)										
34	23.50		0(7)										
35	24.00		0(0)										
36	25.00		0(0)										
37	26.00			0(7)									
38	27.00			0(7)									
39	28.00			0(0)									
40	29.00			0(0)									
41	30.00			0(5)									
42													
43													

Hide

Figure 4: Mode (Function) notation is used. 0 is for major and 5 is for minor mode. 0 is for tonic, 5 is for subdominant and 7 is for dominant function. The first row lists all tonalities (F,C, G,…) in the order of cycle of fifths.

Our analysis model does not presuppose a tonal setup. This is an important difference between the computational model we use and the traditional harmonic analysis. That is to say, we do not use key information if it is present in the MIDI file. In the traditional score-based analysis by an individual, the tonal setup is given globally as accidentals or in the name 'Sonata in D minor'. In our model, however, tonality/key is extracted from local pitch relations in their vertical organizations as chords and horizontal motion of chords as harmonic functions. In that sense, we claim that there is no “ideal'' harmonic analysis of the piece but there are different “harmonic perspectives” as a result of different parametrizations by the user. Furthermore, our computational analysis is not a simulation of the traditional harmonic analysis by a musician working on the score but a new way of looking at harmony as well as a method for harmonic analysis. This also allows questioning some assumptions of the traditional harmonic analysis method. For example, having as few key changes as possible is such a criterion in the analysis of tonal music. On our network, if the tension of going from one tonality to another tonality is very low, having as few key changes as possible would no longer be a criterion.

4.2. Analysis of Local Meters

Similarly, the mathematical model of a local meter is based on all periodic onsets in a composition. Through the model, “weights” are assigned to these onsets and they can be visualized as an interactive graph (Mazzola et al., 2020). Metric weight idea is based on Riemann's observations and insights about meter (Mazzola, 2002). Local meters span a composition just like local tonalities (or keys) span a composition. Meter is the periodicity of onsets in the time domain and key/tonality is periodicity in the pitch domain. Mazzola (2002) defines a *maximal local meter* as the periodic onsets in a composition. While in music theory the definition of meter involves the periodicity of a beat pattern with strong and weak beats, the strength or -in Riemann's terminology- 'weight' of a beat is not a priori but is *generated* in the composition by onsets. A composition's *notated* or *global meter*, such as 4/4 or 3/4, could give information about the meter and impose a pattern of strong and weak beats. However, an analysis of all maximal local meters and their weights is a highly detailed chart of the metric composition of the entire piece.

The figure in Appendix 1 enlists three maximal local meters m_1, m_2, m_3 that span the onsets $o_1, \ldots o_{11}$. We can see that the maximal local meters m_1, m_2 and m_3 have different *periods*. Some of these onsets such as o_5 and o_9 are on two different maximal local meters. The *length* of the maximal local meter m_1 is 3 while the lengths of m_2 and m_3 are 2.

At this point, it should be noted that, while in harmony, periodicity in pitch (as a chain of thirds) is examined; in a local meter, periodicity of onsets is examined. In that aspect, harmonic analysis and metric analysis are similar models, working on two different domains: pitch and time[5].

Mazzola's (2002) metric weight formulation computes a weight graph for each onset where onsets that are within longer maximal local meters or in more than one maximal local meter are assigned higher weights.

$$W_{l,p}(o) = \sum_{m, k(m) \geq l, o \in m} k(m)^p$$

Through this formula, a weight is assigned to an onset o which is at least of length l and is in a maximal local meter m of length $k(m)$. The metrical profile p is the parameter that determines how the length of the maximal local meter contributes to the weight. In the figure in Appendix 1, o_5, o_8, and o_9 have higher weights than the

5 Analysis of onsets to extract maximal local meters is a simple computation but care should be taken to have adequate quantization and strict adherence to musical notation in the symbolic representation of music which in our case is a MIDI file.

remaining onsets. High weight onsets are structurally more important, and in the case of certain genres such as the Baroque genre and popular genres such as rock, pop ballad, etc., these high weight onsets align with the global meter. This means strong beats that are suggested by the notated meter. Fleischer (2002) names this alignment as *metric coherence* and enlists musical examples which have metric coherence. Such a weight analysis allows much more differentiation than the traditional strong-weak beat dichotomy. Recent research has shown benefits of metric weight computation on the style analysis of various music genres (Mazzola, 2002; Fleischer, 2002; Volk, 2009; Mazzola et al., 2020).

4.3. Analysis of Harmony that Uses Local Meters

In music of the common practice period, meter imposes a hierarchy to the perception of harmony and melody. That is to say, we do not hear every musical event as if the event is on the strong beat. In common practice period music, passing and neighbour tones are usually on the weak beats; an authentic cadence usually ends on a strong beat. When we tap our foot or clap our hands with such music, we are typically synchronizing with the strong beats of the meter.

GTTM analyzes musical events and derives the *metrical structure*, a hierarchical structure which is represented by a tree of dots corresponding to beats (that may or may not align with the onsets). The strength of the beat corresponds to its depth in the tree as seen in the figure in Appendix 2. A very counter-intuitive result of GTTM's analysis is the insertion of a dot between the first two notes in the 1st and 2nd measures without a corresponding onset. GTTM's metrical analysis procedure is the sum of metrical well-formedness rules and preference rules (Lerdahl and Jackendoff, 1983) but GTTM does not give an algorithm to derive a metrical structure.

An analysis of the metrical structure using Mazzola's method results in maximal local meters a, b, and c that span every onset in the score. Local meters a and b will have the highest weight; in fact, they span the whole 8-measure long period Mozart K. 331, i, mm. 1-8 (Alpaydin, 2022). The main difference between the two ways of metric analyses is that, while GTTM imposes the notated global meter to infer the metrical hierarchy, in Mazzola's method, maximal local meters are generated from onsets only, and weights are assigned according to the inclusion of onsets in the local meters. It should be noted that this approach is also similar to the harmonic analysis method discussed in the previous section. For common practice period music, global tonality in a score can be inferred from the notated accidentals. Our harmonic analysis does not have any a priori tonal information and tonalities are extracted from the vertical and horizontal organizations of pitches only.

If different onsets have different strengths (or weights) and this creates a metrical structure with a hierarchy, this should also influence the way harmony is perceived. We wanted to inject the perceived metrical hierarchy into the computational analysis of harmony. For this purpose, we analyzed chords that are on the same maximal local meter and compared harmonic progression with metrical proximity to harmonic progression with temporal proximity. Taking metrical proximity as primary also aligns with a well-known auditory perception mechanism in humans: *metrical entrainment*. In tonal music, structurally important chords are usually on the strong beats and embellishing chords are on relatively weaker beats, which is a well-known observation. We describe in detail the new harmonic analysis algorithm that considered metrical information as well and review the results of execution of the new algorithm on a Mozart and a Beethoven composition (Alpaydin, 2022). Temporal proximity and metrical proximity cases differ very little in terms of the resultant harmonic path. This may indicate that harmonic progression is sensible not only locally-and-temporally but metrically as well, on a broader level, an intuition that great composers have. The implementations of harmonic analysis and metric analysis as software can be downloaded and used as plug-in components under Rubato[6].

6 The rubettes for analysis of harmony and meter can be downloaded from http://www.rubato.org/rubettes.html.

Conclusion and Future Directions

Our sample set is limited and we plan to extend our set to other examples of European tonal music such as those of J. S. Bach, Haydn, and other composers of the common practice era. We expect similar results, i.e., alignment of temporal and metrical considerations of harmony to a great extent.

Sophisticated experiments about music cognition is possible through algorithms that analyze complex structures in musical data. Analysis that goes beyond pen and paper analysis on the score or a large number of scores is possible because computers process symbolic information faster than human beings (Mazzola et al., 2020). Use of algorithms and computational power will broaden the horizons of music theory. For example, it is now possible to analyze the semi-cadence structure of all J. S. Bach compositions, the development of all Mozart sonatas, the modulations of Beethoven, or the details of the metric language in a Brahms sonata. Thus, a precise, statistical conclusion can be made. In this way, some features that were felt at the intuitive level when we listened to music before can now be grounded and explained through computational analysis. Generalizations are easier and quicker to make with an objectivity that does not allow vagueness, inconsistency or a subjective conclusion without rationale. With its precision, speed and memory, computer power extends a theorist's mind, but indubitably, it still requires a theorist to propose a thesis, model cognition, and design algorithms.

Hakem Değerlendirmesi *Peer-review*	**Bağımsız** *Externally peer-reviewed.*
Çıkar Çatışması *Conflict of Interest*	**Yazar çıkar çatışması bildirmemiştir.** *The author has no conflict of interest to declare.*
Finansal Destek *Financial Support*	**Yazar çalışması için finansal destek almadığını beyan etmiştir.** *The author declared that this study has received no financial support.*
Teşekkür Açıklaması *Acknowledgements*	**Yazar teşekkür açıklaması bildirmemiştir.** *The author did not provide a statement of acknowledgements.*

References

Agon, C., Assayag, G., Laurson, M., Rueda, C. & Delerue, O. (1999). Computer Assisted Composition at Ircam: PatchWork & OpenMusic. *Computer Music Journal*, 23(3).

Agustín Aquino, O. A. (2011). Extensiones microtonales de contrapunto. (Tesis de Doctorado). Universidad Nacional Autónoma de México, México. Recuperado de <https://repositorio.unam.mx/contenidos/92950>.

Alpaydin, R. &Mazzola G. (2015). A Harmonic Analysis Network. *Musicae Scientiae*, 19(2), 191-213.

Alpaydin, R. (2022, March 20). "Harmonic Analysis with Rhythmical Metrical Weights". (PhD thesis in preprint). https://www.researchgate.net/publication/347555700_Computational_Harmonic_Analysis_with_Rhythmical_Weights_PhD_thesis.

Andreatta, M., Ehresmann, A., Guitart, R. & Mazzola, G. (2013). Towards a categorical theory of creativity. Proceedings of the Conference MCM 2013, McGill University, Montreal. Lecture Notes in Computer Science / LNAI, Springer, 19-37.

Bibby, N. (2006). Tuning and Temperament: Closing the Spiral. In: *Music and Mathematics*, Oxford University Press. 13-28.

Boolos G.,Burges, J.P., &Jeffrey, R.C. (1974). *Computability and Logic*. (1st ed.). Cambridge University Press.

Bresson, J. & Carlos, A. (2010). Processing Sound and Music Description Data using OpenMusic. Proc. International Computer Music Conference, New York, USA.

Burkholder J.P., Grout D.J. & Paisca J.V. (2014). *A History of Western Music*. New York: WWNorton & Co.

Caplin, W.E. (1998). *Classical Form: A Theory of Formal Functions for the Instrumental Music of Haydn, Mozart, and Beethoven.* New York: Oxford University Press.

Di Guigno, G., & A. Gerzso. (1986). La station de travail musical 4X. IRCAM Technical Report. Paris: IRCAM.

Edwards, C.H. (1982). *The Historical Development of the Calculus*. Springer Study edition. (2nd ed.). Springer-Verlag.

Fiore, T. & Noll T. (2018). Voicing Transformations of Triads, *SIAM J. Appl. Algebra Geometry*, 2(2), 281–31.

Fleischer, A. (2002). A Model of Metric Coherence. In *Proceedings of the 2nd international conference understanding and creating music*. Caserta, Italy. https://webspace.science.uu.nl/~fleis102/Casertaletter.pdf.

Forte, A. (1977). *The Structure of Atonal Music*. Yale University Press.

Hiller, L.A. & Issacson, L.M.. (1957). Illiac Suite for String Quartet. New Music Edition. Theodore Presser Company, Bryn Mawr, Pa.

Hilton, P. (1984). Cryptanalysis in World War II —and Mathematics Education, *The Mathematics Teacher*, 77(7), 548-552. https://pubs.nctm.org/view/journals/mt/77/7/article-p548.xml

Hirschberg, A., Gilbert, J., Msallam, R., &Wijnands, A.P.J. Shock Waves in Trombones. *Journal of the Acoustical Society of America*, 99(3), 1754-1758.

Krumhansl, C.L. (1990). *Cognitive Foundations of Musical Pitch*. Oxford University Press, New York.

Lerdahl, F.&Jackendoff, R. (1983). *A Generative Theory of Tonal Music*. Cambridge, MA: MIT Press.

Lewin, D. (1982). A Formal Theory of Generalized Tonal Functions. *Journal of Music Theory*, 16 (1), 23-60.

Lewin, D. (1993). *Musical Form and Transformation: Four Analytic Essays*. Yale University Press: New Haven, Connecticut.

Mazzola, G.(1985). *Gruppen und Kategorien in der Musik*. Heldermann, Berlin.

Mazzola, G. (2002). *The Topos of Music*. Basel-Boston-Berlin: Birkhauser Verlag.

Mazzola, G. (2011). Oniontology. *In Musical Performance*. Computational Music Science. Springer, Berlin, Heidelberg. 21-34.

Mazzola, G., Alpaydin, R. &Heinze, W. (2020). Software-based Experimental Music Theory: Rubato's Metrorubette for Brahms' Sonata Op. 1. In: The *Future of Music - Towards a Computational Musical Theory of Everything*. Springer. 49-61.

Milmeister, G. & Mazzola, G. (2007). Rubato Composer Software. https://www.rubato.org/rubatocomposer.html.

Milmeister, G. (2007). Overview of Music Theories. In: *The Rubato Composer Music Software*, Springer, 3-6.

Montiel, M. (2011). The Rubato Composer Software Concept for Learning Advanced Mathematics. In Emilio Lluis-Puebla & Octavio Agustín-Aquino (Eds), *Memoirs of the Fourth International Seminar on Mathematical Music Theory*. Electronic Publications of the Mexican Mathematical Society, 77-96.

Noll, T. (1997). Morphologische Grundlagen der abendländischen Harmonik. Brockmeyer, Bochum.

Noll, T. &Garbers, J. (2001). Harmonic path analysis. International Seminar on Mathematical and Computational Musik TheoryAt: SauenVolume: Mazzola, Guerino / Noll, Thomas / Lluis-Puebla, Emilio (eds): *Perspectives in Mathematical and Computational Music Theory*.

Agustín Aquino, O. A.&Mazzola, G. (2019). Contrapuntal Aspects of the Mystic Chord and Scriabin's Piano Sonata No. 5. In Montiel M., Gomez-Martin F., Agustín-Aquino O. (Eds), *Mathematics and Computation in Music*. MCM 2019. Lecture Notes in Computer Science, vol 11502. Springer, Cham.

Parncutt. R. (2008). Interdisciplinarity in JIMS. *Journal of Interdisciplinary Music Studies*, (2) (1&2). http://musicstudies.org/wp-content/uploads/2017/01/interdisciplinarity_JIMS.pdf.

Roads, C., & Mathews, M. (1980). Interview with Max Mathews. *Computer Music Journal*. 4(4), 15–22. https://doi.org/10.2307/3679463.

Schedl, M., Gómez, E. & J. Urbano.Music. (2014). Music Information Retrieval:Recent Developments and Applications. In *Foundations and Trends In Information Retrieval*, 8(2-3), 127–261. http.//doi.org/10.1561/1500000042.

Sipser, M. (2007). Context-Free Grammars, In: *Introduction to the Theory of Computation*, (2nd ed.), Thomson Course Technology, 100-104.

Temperley, D. (1997). An Algorithm for Harmonic Analysis. *Music Perception*, 15(1), 31–68. https://doi.org/10.2307/40285738

Temperley, D. (2002). A Bayesian approach to key-finding. In C. Anagnostopoulou, M. Ferrand & A. Smaill (Eds.), ICMAI 2002, LNAI 2445. Berlin, Germany, Springer-Verlag. 195-206.

Thalmann, F. (2014). Gestural composition with arbitrary musical objects and dynamic transformation networks. Retrieved from the University of Minnesota Digital Conservancy. https://hdl.handle.net/11299/165105.

Tzanetakis, G.,A Kapur, WA Schloss& Wright, M. (2007). Computational ethnomusicology, *Journal of interdisciplinary music studies*, 1(2), 1-24.

Turing, A.M. (1939). Systems of Logic Based on Ordinals. Proceedings of the London Mathematical Society, 2-45 and 161-228.

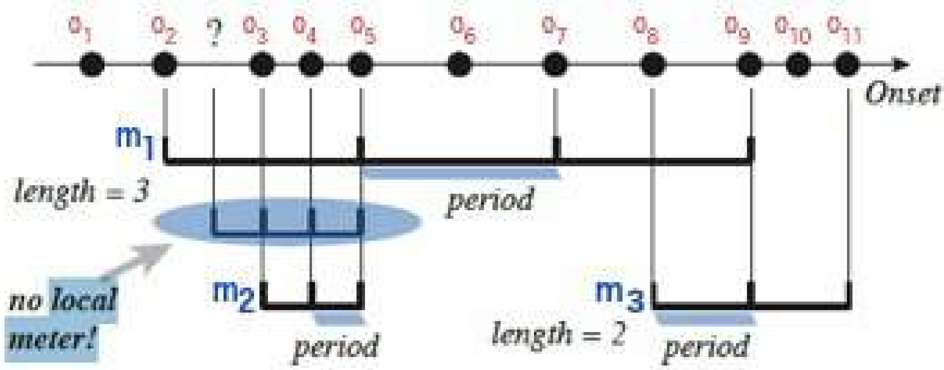
Turing Machine Visualization. (2022, March 12). https://turingmachine.io/.

Turing Machine Physical Realization (2022, March 12). https://hyperglitch.com/articles/turing-machine.

Volk, A. (2009). Applying Inner Metric Analysis to 20th Century Compositions. In *Communications in Computer and Information Science*. Berlin Heidelberg: Springer. 204-210.

Weyl, H. (1983). *Symmetry*. Princeton University Press.

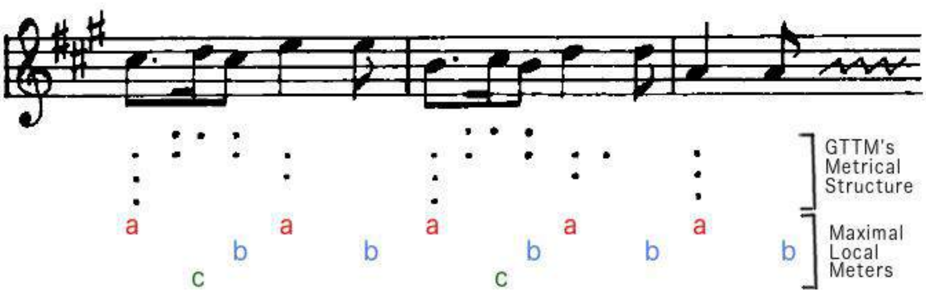
APPENDICES:



APPENDIX 1:

Onsets $o_1,\ldots,o_{11}$ and the maximal local meters they belong to. The drawing is adapted from Mazzola (2002).

APPENDIX 2:



The first two and a half measures of Mozarf K. 331, i Both GTTM`s analysis and maximal local meters are given under the score.